Non-perturbative analysis of the gravitational energy in Hořava Theory

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Abstract

We perform a non-perturbative analysis of the constraints of the Hořava Gravitational theory. In distinction to Einstein gravity the theory has constraints of the first class together with second class ones. We analyze the consequences of having to impose second classes constraints at any time in the quantum formulation of the theory. The second class constraints are formulated as strongly elliptic partial differential equations allowing a global analysis on the existence and uniqueness of the solution. We discuss the possibility of formulating the theory in terms of a master action with first class constraints only. In this case the Hořava theory would correspond to a gauged fixed version of the master theory. Finally we obtain , using the non-perturbative solution of the constraints, the explicit expression of the gravitational energy. It is, under some assumptions, always positive and the solution of Hořava field equations at minimal energy is the Minkowski metric.

1 Introduction

Recently, Hořava [1] proposed a gravitational theory with Lifshitz-like anisotropic scaling at short distances:

$$t \to b^z t, \quad x \to bx.$$

The formulation breaks the relativistic symmetry at short distances with the idea of regaining it at large distances. The benefit would be to obtain a power counting renormalizable theory of gravity. The anisotropic scaling between time and space allows to include in the action high enough spatial derivatives which contribute to the interactions and to the propagators improving the UV properties of the theory. In order to obtain a renormalizable theory z should be equal to the number of spatial dimension and all terms compatible with the gauge symmetry should be included in the action.

In distinction to General Relativity, Hořava Gravity is restricted not only by first class contraints but also by second class constraints which are potentially dangerous because they may introduce non-localities in the quantum formulation of the theory. Directly related to this analysis and an important aspect of any gravitational theory is to determine the gravitational energy and to established its positivity. The positive mass theorem plays a fundamental role in General Relativity. It has firstly proved in [2, 3, 4] that for asymptotically flat space-times the total energy-momentum in General Relativity is well defined, it is greater or equal to zero and it vanishes for flat space-time. We will discuss the existence and uniqueness of the solution to the constraints and will prove the positivity of the gravitational energy for the Hořava theory in the large distance regime. Moreover the expression for the gravitational energy we will obtain remains valid even when one includes all the interacting terms corresponding to the $z = 3$ Hořava theory.

2 The Hořava action

Hořava theory is formulated on a foliated manifold $\mathcal{M} = \Sigma \times \mathbb{R}$ where Σ is a three dimensional Riemann manifold which we will assume to be complete, connected and asymptotically flat.

The theory is expressed in terms of an ADM formulation where the metric is given by

$$ds^2 = \left(-N^2 + N_i N^i\right) dt^2 + 2N_i dx^i dt + g_{ij} dx^i dx^j. \tag{1}$$

The extrinsic curvature of Σ, denoted K_{ij}, satisfies

$$K_{ij} = \frac{1}{2N} \left(\dot{g}_{ij} - \nabla_i N_j\right) \tag{2}$$

where the lapse N is assumed to be different from zero.

The Hamiltonian of Hořava theory is generically given by

$$\int dt d^3x \sqrt{g} N \left(G^{ijkl} K_{ij} K_{kl} + V\left(g, N\right)\right) \tag{3}$$

where

$$G^{ijkl} = \frac{1}{2} \left(g^{ik} g^{jl} + g^{il} g^{jk} - \lambda g^{ij} g^{kl}\right), \tag{4}$$

λ is a dimensionless parameter which may be included because each term is separately a tensor under spatial diffeomorphism the gauge symmetry of the theory. More precisely the above action is invariant under diffeomorphism on Σ and reparametrization on t:

$$\widetilde{t} = f(t), \quad \widetilde{x} = g(x). \tag{5}$$

The symmetry (5) may be enlarged by considering anisotropical conformal gauge transformations, which are relevant in the formulation of conformal gravity [Hořava, 2012][5]. We will only be concerned with transformations (5).

In the potential $V(g, N)$ one should include all possible local, up to $z = 3$ terms, compatible with the gauge symmetries of the theory.

Typical $z = 3$ terms are $\nabla_k R_{ij} \nabla^k R^{ij}, R\Delta R, R^{ij}\Delta R_{ij}$ which contribute to the interactions but also modify the propagators improving the UV behaviour of the theory. Other $z = 3$ terms such as $R^3, RR_{ij}R^{ij}$ are pure interacting terms.

3 The hamiltonian in the large distance regime

The most general Hamiltonian describing the large distance regime in Hořava theory is given by [6, 7]

$$H = \int d^3x \left(N\mathcal{H} + N_i\mathcal{H}^i + \sigma\phi \right) + E_{ADM} - 2\alpha\phi_N \tag{6}$$

$$\mathcal{H} \equiv G_{ijkl}\frac{\pi^{ij}\pi^{kl}}{\sqrt{g}} + \sqrt{g}\left(-R + \alpha\left(2\nabla_i a^i + a_i a^i \right) \right) \tag{7}$$

$$\mathcal{H}^i \equiv -2\nabla_j \pi^{ji} + \phi\partial^i N \tag{8}$$

$$\phi_N \equiv \oint d\Sigma_i \partial_i N, \tag{9}$$

where we have assumed that $\lambda \neq \frac{1}{3}$. For $\lambda = \frac{1}{3}, G^{ijkl}$ is not invertible. The theory with $\alpha = 0$ for any value of λ is equivalent to General Relativity [8, 9]. The terms which depends on N and their derivatives in the potential were introduced in [10].

As in General Relativity the ADM energy in included as the boundary term. This is necessary in order to obtain the equation of motion under variations δg_{ij} asymptotically of order $O(r^{-1})$ [11]. Similarly, the flux of N at spatial infinity, ϕ_N, cancels a non zero contribution coming from $\delta(2\alpha N\nabla_i a^i)$ for variations of N with asymptotic decay $\delta N = O(r^{-1})$.

$\mathcal{H}^i = 0$ is a first class constraint related to the generator of spatial diffeomorphisms. $\mathcal{H} = 0$ is a second class constraint. In order to preserve it we obtain a PDE for the Lagrange multiplier σ,

$$\partial_i\left(N\sqrt{g}g^{ij}\partial_j\left(\sigma/N\right)\right)=\partial_i\left(-\gamma N^2\nabla^i\pi+N^2G^i_{klm}\pi^{lm}\right). \tag{10}$$

The Dirac procedure ends at this stage.

4 The non-perturbative analysis of the constraints

In order to analize the constraints $\mathcal{H}=0$ and the equation for the Lagrange multiplier we consider the conformal transformation

$$\begin{array}{l} g_{ij}=e^{\varphi}\widehat{g}_{ij} \\ \pi^{ij}e^{-\varphi}\widehat{\pi}^{ij} \end{array} \tag{11}$$

where the Riemannian metric $\widehat{g}_{ij}$ satisfies $\det\widehat{g}_{ij}=1$.

We also consider the decomposition of $\pi^{ij}=\widehat{\pi}^{ij}_T+\frac{1}{3}\widehat{g}^{kj}\widehat{\pi}$ where $\widehat{g}^{ij}$ is the inverse of $\widehat{g}_{ij}$ and $\widehat{\pi}^{ij}_T$ is traceless: $\widehat{g}_{ij}\widehat{\pi}^{ij}=0$.

The transformation $(g_{ij},\pi^j)\to\left(\widehat{g}_{ij},\widehat{\pi}^{ij}_T,\varphi,\widehat{\pi}\right)$ is canonical:

$$\int_\Sigma d^3x\pi^{ij}\dot{g}_{ij}=\int_\Sigma d^3x\left(\widehat{\pi}^{ij}_T\dot{\widehat{g}}_{ij}+\widehat{\pi}\dot{\varphi}\right). \tag{12}$$

In the new variables the constraint $\mathcal{H}=0$ becomes

$$\partial_i\left(e^{\beta\varphi}\widehat{g}^{ij}\partial_j e^{\frac{\xi}{2\alpha}}\right)+Ge^{\frac{\xi}{2\alpha}}=0 \tag{13}$$

where

$$G=-\frac{e^{\beta\varphi}}{4\alpha}\left[e^{-2\varphi}\left(\widehat{\pi}^{ij}_T\widehat{\pi}_{Tij}+(3-9\lambda)^{-1}\right)\widehat{\pi}^2-\widehat{R}-\beta\partial_i\varphi\partial^i\varphi\right] \tag{14}$$

and $\xi\equiv\alpha\ln|N|+\varphi$.

We notice that G depends on the canonical pairs $\left(\widehat{g}_{ij},\widehat{\pi}^{ij}_T\right)$ and $(\varphi,\widehat{\pi})$. It does not depend on N. Equation (13) is a linear partial differential equation on $e^{\frac{\xi}{2\alpha}}$. It is strongly elliptic. It is convenient to consider $e^{\frac{\xi}{2\alpha}}=1+u$ and to define a suitable space of functions for u.

We can prove the following proposition [12]:

Proposition 1 *Given an asymptotically flat Riemannian manifold with C^∞ metric g_{ij} and momenta π^{ij}, with the asymptotic behaviour*

$$\begin{array}{l} g_{ij}=\delta_{ij}+O(r^{-1}) \\ \pi^{ij}=O(r^{-2}), \end{array}$$

then the asymptotic solution for (13) exists, it is C^∞ and the asymptotic behaviour is

$$u=O(r^{-1}).$$

We are then motivated to introduce the space $\widehat{C}^1$:

$$\widehat{C}_1(\Sigma) = \left\{ u \in C^1(\Sigma) : u = O(r^{-1}) \text{ when } r \to \infty \right\}.$$

For u and v in $\widehat{C}_1(\Sigma)$ we define the bilinear functional

$$(u,v) = \int_\Sigma d^3x \left(e^{\beta\varphi} \widehat{g}^{ij} \partial_i u \partial_j v + Guv \right).$$

If $G \geq 0$ on Σ then (u,v) defines an internal product in $\widehat{C}_1(\Sigma)$. We denote $\widehat{\mathcal{H}}^1$ the Hilbert space obtained by the completion of $\widehat{C}_1(\Sigma)$ with respect to the norm induced by the above internal product.

The assumption $G \geq 0$ will be essential to prove existence and uniqueness for the solution of the constraint. If G is not positive the operator

$$\mathcal{O} \equiv -\partial_i \left(e^{\beta\varphi} \widehat{g}^{ij} \partial_j \cdot \right) + G \cdot$$

may have a nontrivial kernel K. In that case the solution u to the constraint

$$\mathcal{O}u = -G \tag{15}$$

exists if and only if G is orthogonal to K. Even if this condition is satisfied the solution would not be unique.

We can prove the following propositions [12].

Proposition 2 *Given a complete, connected, asymptotically flat Riemannian manifold Σ and momenta π^{ij} satisfying $G \geq 0$, there always exist in $\widehat{\mathcal{H}}^1$ a unique weak solution to the constraint (15).*

Proposition 3 *Under the previous assumptions and considering a C^∞ metric g_{ij} and momenta π^{ij}, the weak solution is C^∞.*

Proposition 4 *Under the previous assumptions the solution satisfies $1 + u \geq 0$. Hence we may identify*

$$e^{\frac{\xi}{2\alpha}} \equiv 1 + u \geq 0.$$

Proposition 5 *Under the previous assumptions the solution for the Lagrange multiplier σ exists and is unique. The asymptotic behaviour is*

$$\sigma = O(r^{-2})$$

when $r \to \infty$.

These propositions prove under above assumptions, the existence and uniqueness of the solution to the constraint $\mathcal{H} = 0$ and of the equation for the Lagrange multiplier.

We can finally obtain the following results concerning the gravitational energy of the Hořava gravity.

The gravitational energy is given by the flux of the ξ field:

$$E = -2 \oint d\Sigma_i g^{ij} \partial_j \xi.$$

Under the previous assumptions and for $\alpha \leq 0$ the gravitational energy E is positive.

The energy E has a minimum for $\xi = 0$. At the minimum and using the field equations for Hořava gravitational theory we obtain

$$\begin{array}{c} g_{ij} = \delta_{ij}, \pi^{ij} = 0 \\ N = 1 \end{array} \tag{16}$$

in the gauge $N^i = 0$.

It is important to distinguish this space-time metric from one obtained by performing an anisotropic conformal transformation on a Lifchitz space-time. If fact, the field equations for the action we are considering are invariant under an isotropic conformal transformation (exactly the same one that leave invariant Einstein equations for General Relativity). The metrics are equal but the symmetries of the underlying spaces are different.

5 Conclusions

We presented a non-perturvatibe analysis of the existence and uniqueness of the second class contraints of the Hořava Gravity in the large distance regime where $z = 3$ interacting terms and their contributions to the propagator are not relevant. The action we considered includes all the $z = 1$ possible contributions (without the cosmological term). We provide a sufficient condition ensuring the existence and uniqueness of the solution to the constraints. After the elimination of : a) the lapse N, from the solution for the ξ field, and its conjugate momentum from the second class contraints, b) the gauge field N^i and the longitudinal part of π^{ij} from the first class constraint, we are then left with the physical degrees of freedom in terms of the conjugate pairs $\left(\pi_T^{ij}, \hat{g}_{ij}\right)$ and $(\hat{\pi}, \varphi)$. There is one additional degree of freedom with respect to General Relativity. The interesting point of our presentation is that the gravitational energy of the theory is directly expressed as the flux of the ξ field whose solution is obtained directly from the second class constraint $\mathcal{H} = 0$. We gave a sufficient condition ensuring the positivity of the gravitational energy for Hořava theory.

Although the second class contraints can be qualitatively solved and the solution renders a positive gravitational energy still the goal of a renormalizable field theory has

not been reached. The reason is that the complicated (when $z = 3$ contributions are included) second class constraints have to be imposed at any time and the possibility to explicitly solve them without introducing non-localities is far for been obtained. An alternative approach to handle this problem arises from the structure of the metric of Poisson bracket of the second class constraint. Their contribution to the measure of the path integral is through the square root of its determinant. In general, even with the inclusion of $z = 3$ terms in the hamiltonian, the square root reduces only to the bracket $\{\mathcal{H}, \phi\}$. This contribution is similar to the one of a theory with first class constraint only. In that case the bracket is evaluated between the first class constraint and its gauge fixing condition. This suggest the existence of a master action with first class constraints only from which the Hořava theory would arise as a gauge fixed of the master theory. In that case a BRST quantization would then solve the problem.

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